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EXPONENTIAL SHOCK AND RAREFACTION WAVES

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ABSTRACT

The problem of a gas which is being heated at an exponentially increasing rate, and which is confined by a plane wall of unheated material, is solved to find the form of the rarefaction wave in the gas, and the shock wave in the wall. The results have been given in graphical form in another report, LA 10, by R. Davis and S. Frankel.

Our problem is to determine what happens when a gas contained by a plane wall of matter is heated at an exponentially increasing rate. As the pressure rises, the wall is pushed outward, a rarefaction wave runs back into the gas, and a shock wave moves forward into the wall. For high enough pressures the heating of the wall by the shock wave is sufficient to vaporize the wall material; we shall suppose the pressure so high that the internal energy before the arrival of the shock wave is negligible compared to the shock wave heating. The only properties of the wall which then concern us are its initial density, and the relation between internal energy, U, and pressure, p, for the vaporized material. This will be taken of the form $p = (\gamma - 1)U$ with γ constant. A similar pressure-energy relation will be used for the gas confined by the wall.

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THE EXPONENTIAL SHOCK WAVE

Since the interface between wall and gas may be expected to move exponentially, we first consider the shock wave produced by an exponentially accelerated piston.

Let x be the original position of a given element of a mass in the wall, X(xt) be its position at time t. The equation of motion is

$$\rho_1 \ddot{X} = - \frac{\partial p}{\partial x}, \tag{1}$$

where ρ_1 is the initial density of the material.

The density at any later time is given by

$$\frac{\rho}{\rho_1} = \frac{1}{\partial X/\partial x} \quad . \tag{2}$$

At the wall-piston interface, x = 0, we have the boundary condition

$$X\ (ot) = X_0 e^{\alpha t} \ . \tag{3}$$

The boundary conditions at the shock front are

$$\rho_f = \frac{\gamma+1}{\gamma-1} \rho_1 \tag{4}$$

$$V^2 = \frac{\gamma+1}{2} \frac{p_f}{\rho_1} \tag{5}$$

$$u^2 = \frac{2}{\gamma+1} \frac{p_f}{\rho_1} \tag{6}$$

where ρ_f and p_f are the density and pressure at the front, V is the shock wave velocity, u is the material velocity at the front. Behind the front the material is compressed adiabatically:

$$\frac{p}{\rho^{\gamma}} = S(x). \tag{7}$$

The adiabicity of the expansion is expressed by the fact that S is not a function of t; the dependence of S on x is determined by the entropy change imparted by the shock wave.

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Teller has shown that these equations can be solved by a similarity transformation of the form

$$x = X_1 e^{\alpha t}\varphi(\xi) \quad , \tag{8}$$

$$p = p_1 e^{2\alpha t} f(\xi) \quad , \tag{9}$$

$$\xi = \frac{x}{X_1 e^{\alpha t}} \quad , \varphi(1) = 1, \; f(1) = 1 \tag{10}$$

The variable ξ is the relative position from back to front of the shock wave, $\xi = 0$ at the piston, $\xi = 1$ at the shock front. The position of the shock front is $x = X = X_1 e^{\alpha t}$, the pressure at the front is $p_1 e^{2\alpha t}$.

Using (8) and (9), (1) becomes

$$\xi^2\varphi'' - \xi\varphi' + \varphi = -\frac{p_1}{\rho_1 X_1^2 \alpha^2} f' \quad , \tag{11}$$

the prime denoting differentiation with respect to ξ.

(2) takes the form

$$\frac{\rho_1}{\rho} = \varphi' \quad , \tag{12}$$

whence (4) gives the boundary condition

$$\varphi'(1) = \frac{\gamma - 1}{\gamma + 1} \quad . \tag{13}$$

The left hand side of (7) is now

$$\frac{p}{\rho^\gamma} = \frac{p_1}{\rho_1^\gamma} f \varphi'^\gamma e^{2\alpha t} \quad ,$$

and S(x) must be the value of this quantity immediately after the shock wave hits the point x, i.e.,

$$S(x) = \frac{p_1}{\rho_1^\gamma} f(1)(\varphi'(1))^\gamma \left(e^{2\alpha t}\right)_{x = X_1 e^{\alpha t}} = \left(\frac{\gamma-1}{\gamma+1}\right)^\gamma \frac{p_1}{\rho_1^\gamma}\left(\frac{x}{X_1}\right)^2$$

Thus (7) becomes

$$f\varphi'^\gamma = \left(\frac{\gamma-1}{\gamma+1}\right)^\gamma \xi^2 \quad . \tag{14}$$

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The shock wave velocity equation, (5), gives

$$V^2 = \frac{\gamma+1}{2}\,\frac{p_1}{\rho_1}\,e^{2\alpha t} = \alpha^2 X_1^2 e^{2\alpha t},$$

or

$$\frac{p_1}{\rho_1 \alpha^2 X_1^2} = \frac{2}{\gamma+1}. \tag{15}$$

This result can be checked by direct integration of the velocity equation,

$$X_1 e^{\alpha t} = \int_{-\infty}^{t} V\,dt = \sqrt{\frac{\gamma+1}{2}\,\frac{p_1}{\rho_1}}\int_{-\infty}^{t} e^{\alpha t} = \sqrt{\frac{\gamma+1}{2}\,\frac{p_1}{\rho_1}}\;\frac{e^{\alpha t}}{\alpha}.$$

Finally we can verify that the remaining shock wave condition, (6), is satisfied: the material velocity at any point is

$$\dot{X} = \alpha X_1 (\varphi - \xi\varphi')e^{\alpha t},$$

so

$$u = \dot{X}(1) = \alpha X_1 (1 - \varphi'(1))e^{\alpha t} = \frac{2}{\gamma+1}\,\alpha X e^{\alpha t} = \frac{2}{\gamma+1}\,V,$$

in agreement with (5) and (6).

The similarity transformation (8) and (9) is thus compatible with all the equations and boundary conditions. We are left with two equations, (11), which, using (15), can now be written

$$\xi^2\varphi'' - \xi\varphi' + \varphi = -\frac{2}{\gamma+1}\,f', \tag{16}$$

and (14), which together determine the two functions f and φ, and the detailed properties of the shock wave. Since these two equations form a second order system, and two boundary conditions are given in (10), the solution is uniquely determined. The third boundary condition (13) was used in deriving (14), and is automatically satisfied in virtue of (14) and (10). It is notable that the form of the shock wave depends only on γ and is independent of all other features of the problem.

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It follows from (14) that, since the pressure and therefore f remains finite at the piston, the density becomes infinite as $\xi^{-2/\gamma}$. The reason for the appearance of this singularity is the neglect of the entropy of the material before the shock wave hits it. It is clear from (14) that inclusion of the initial entropy would round off the density to the value appropriate to an adiabatic compression from the initial to the final pressure. Similarly, the temperature at the piston maintains its initial value, zero.

We have expressed the solution in terms of the distance moved by the shock front, rather than the distance moved by the piston. The ratio between these distances is given by $\frac{X_0}{X_1} = \varphi(0)$, and can be determined only by integration of the equations. Once X_1 is found in this way, the front pressure is given by (15).

The equations (14) and (16) can be solved by a straightforward numerical integration, starting at $\xi = 1$ since the boundary conditions are given at this end. Near $\xi = 1$, φ and f are represented by the power series

$$\varphi = 1 - \frac{\gamma_1 - 1}{\gamma_1 + 1}\left[(1-\xi) - \frac{3}{\gamma_1+1}(1-\xi)^2 + \ldots\ldots\right]$$

$$f = 1 + \frac{2(2\gamma_1 - 1)}{\gamma + 1}(1-\xi) + \ldots\ldots$$

A useful check on the numerical integration can be obtained from the energy and momentum conservation laws. The total momentum of the wave is

$$P = \rho_1 \int \dot{X}\, dx = \alpha \rho_1 X_1^2 e^{2\alpha t} \int_0^1 (\varphi - \xi\varphi')\, d\xi ,$$

whence, since $P = p_{x=0}$, we obtain the relation

$$f(0) = (\gamma+1)\int_0^1 (\varphi - \xi\varphi')\, d\xi = (\gamma+1)\left[2\int_0^1 \varphi\, d\xi - 1\right]. \qquad (17)$$

The energy is

$$E = \rho_1 \int \left[\frac{1}{\gamma-1}\frac{p}{\rho} + \frac{1}{2}\dot{X}^2\right] dx$$ [redacted]

$$= \alpha^2 X_1^3 \rho_1 e^{3\alpha t} \int_0^1 \left[\frac{1}{\gamma-1} f \varphi' + \frac{1}{2}(\varphi - \varphi'\xi)^2\right] d\xi ,$$

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and its rate of change is $\dot{E} = (p\dot{X})_{x=0}$, which gives:

$$f(0)\varphi(0) = \frac{3}{2}\int_0^1 \left[\frac{2}{\gamma-1} f\varphi' + \frac{\gamma+1}{2}(\varphi - \varphi'\xi)^2\right] d\xi. \tag{18}$$

THE RAREFACTION WAVE

In the heated gas we suppose the energy generated per gram per second is $\frac{dQ}{dt} = \varepsilon\, e^{2\alpha t}$. The expansion is no longer adiabatic, in place of (7) we have

$$dQ = du + pdV \tag{19}$$

where $V = \frac{1}{\rho}$ is the specific volume, $u = UV$ is the internal energy/gram. With $p = (\gamma_1 - 1)U$, (19) becomes $\frac{1}{\gamma_1 - 1}(Vdp + \gamma_1 pdV) = dQ$, or

$$V\dot{p} + \gamma_1 p\dot{V} = (\gamma_1 - 1)\varepsilon\, e^{2\alpha t}. \tag{20}$$

The equation of motion (1), and the matter continuity equation (2) are now written

$$\rho_{-1}\ddot{X} = -\frac{\partial p}{\partial x}, \tag{21}$$

$$\frac{\rho}{\rho_{-1}} = 1 \Big/ \frac{\partial X}{\partial x}, \tag{22}$$

where ρ_{-1} is the initial density of the gas.

In the gas x is taken negative, $x = 0$ being, as before, the gas-wall interface, and the similarity transformation takes the form

$$X = X_{-1}\, e^{\alpha t}\varphi(\xi) \tag{23}$$

$$p = p_{-1}\, e^{2\alpha t} f(\xi) \tag{24}$$

$$\xi = \frac{x}{X_{-1} e^{\alpha t}}, \quad \varphi(-1) = -1, \quad f(-1) = 1. \tag{25}$$

ξ now runs from -1 at the back of the rarefaction wave, to 0 at the interface. At the back of the wave $x = X = -X_{-1}\, e^{\alpha t}$, $p = p_{-1}\, e^{2\alpha t}$.

The boundary conditions at the back of the wave are the continuity of ρ and p. Since $\frac{\rho_{-1}}{\rho} = \varphi'$, the density condition is

$$\varphi'(-1) = [illegible] \tag{26}$$

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The pressure in the gas not yet reached by the rarefaction wave is

$$p = (\gamma_1 - 1)U = (\gamma_1 - 1)\int_{-\infty}^{t} \varepsilon \rho_{-1} e^{2\alpha t}\,dt = \frac{(\gamma_1 - 1)\varepsilon \rho_{-1}}{2\alpha} e^{2\alpha t},$$

so continuity of pressure demands

$$p_{-1} = \frac{(\gamma_1 - 1)\varepsilon \rho_{-1}}{2\alpha}. \tag{27}$$

The boundary conditions at the interface are that the pressure must be continuous and the displacement must be the same for shock and rarefaction waves.

In place of (11) we now have

$$\xi^2 \varphi'' - \xi \varphi' + \varphi = -\frac{p_{-1}}{\rho_{-1} X_{-1}^2 \alpha^2} f', \tag{28}$$

while (20) gives

$$(2f - \xi f')\varphi' - \gamma_1 \xi f \varphi'' = \frac{(\gamma_1 - 1)\varepsilon \rho_{-1}}{\alpha p_{-1}}, \tag{29}$$

which can be rewritten

$$-\frac{d}{d\xi}\left(\frac{f \varphi'^{\gamma_1}}{\xi^2}\right) = \frac{(\gamma_1 - 1)\varepsilon \rho_{-1}}{\alpha p_{-1}} \frac{\varphi'^{\gamma_1 - 1}}{\xi^3}. \tag{30}$$

For $\varepsilon = 0$, (30) integrates to the form (14). Using (27),

$$-\frac{d}{d\xi}\left(\frac{f \varphi'^{\gamma_1}}{\xi^2}\right) = \frac{2\varphi'^{\gamma_1 - 1}}{\xi^3}. \tag{31}$$

A relation analogous to (15) can be obtained by comparing (28) and (29) at $\xi = -1$. In virtue of (25) and (26), (28) reduces to

$$\varphi''(-1) = -\frac{p_{-1}}{\rho_{-1} X_{-1}^2 \alpha^2} f'(-1), \tag{32}$$

while (29) gives

$$\varphi''(-1) = -\frac{1}{\gamma_1} f'(-1). \tag{33}$$

Thus

$$\frac{p_{-1}}{\rho_{-1} X_{-1}^2 \alpha^2} = \frac{1}{\gamma_1}, \tag{34}$$

which checks that the rarefaction wave moves with the velocity of sound:

$$X_{-1} e^{\alpha t} = \int_{-\infty}^{t} \sqrt{\gamma_1 \frac{p}{\rho_{-1}}}\,dt = \sqrt{\frac{\gamma_1 p_{-1}}{\rho_{-1}}} \int_{-\infty}^{t} e^{\alpha t}\,dt = \sqrt{\frac{\gamma_1 p_{-1}}{\rho_{-1}}}\,\frac{e^{\alpha t}}{\alpha}.$$

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(28) thus becomes

$$\xi^2\varphi'' - \xi\varphi' + \varphi = \frac{1}{\gamma_1} f. \tag{35}$$

The two functions f and φ are now determined by (31) and (35). Similar equations have been obtained by Dirac, but with a different form of energy-generation law. In analogy with the shock wave we have three boundary conditions, given by (25), (26), at the back of the wave. Although we have a third-order system of equations these three boundary conditions do not fix a solution, since, as can be seen from (32), (33), (34), the point $\xi = -1$ is a singular point of the equations: $\varphi''(-1)$ and $f'(-1)$ are not determined, but only their ratio. It then remains possible to fit one boundary condition at the interface. Since both pressure and displacement must be continuous at the interface, the system at first sight seems overdetermined. But it is apparent from (15) that the shock wave solution does not determine both p and X at the interface, but only the ratio $\frac{p}{X^2}$. The boundary condition at $\xi = 0$ thus takes the form

$$\frac{p_{-1} f(-0)}{X_{-1}^2\varphi(-0)^2} = \frac{p_1 f(+0)}{X_1^2\varphi(+0)^2},$$

or, using (15) and (34)

$$\frac{\rho_{-1}}{\gamma_1}\,\frac{f(-0)}{\varphi(-0)^2} = \frac{2\rho_1}{\gamma+1}\,\frac{f(+0)}{\varphi(+0)^2}. \tag{36}$$

THE SOLUTION NEAR $\xi = -1$

Since numerical integration becomes difficult at a singular point, it is desirable to obtain an analytic expression for the solution near $\xi = -1$.

Write $\varphi = \xi + \chi$. The boundary conditions (25), (26) require

$$\underset{(-1)}{\chi(0)} = \underset{(-1)}{\chi'(0)} = 0, \tag{37}$$

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and (35) becomes

$$\xi^2 x'' - \xi x' + x = -\frac{1}{\gamma_1} f . \tag{38}$$

Integrating (31), and choosing the constant of integration to satisfy the boundary conditions,

$$\frac{f\varphi'^{\gamma_1}}{\xi^2} = 1 - 2\int_{-1}^{\xi} \frac{\varphi'^{\gamma_1-1}}{\xi^3}\, d\xi , \tag{39}$$

or, writing $\xi = -1+\eta$,

$$\frac{f(1+x')^{\gamma_1}}{(1-\eta)^2} = 1 + 2\int_0^{\eta} \frac{(1+x')^{\gamma_1-1}}{(1-\eta)^3}\, d\eta . \tag{40}$$

In virtue of (37) x' may be considered of order η ; the integral on the right is then

$$1+2\int_0^{\eta} \frac{1+(\gamma_1-1)x'}{(1-\eta)^3}\, d\eta = \frac{1}{(1-\eta)^2} + 2(\gamma_1-1)x + O(\eta^2),$$

and (40) gives

$$f = 1-\gamma_1 x' + \frac{\gamma_1(\gamma_1+1)}{2}x'^2 + 2(\gamma_1-1)x + O(\eta^3),$$

$$f' = -\gamma_1\left(1-(\gamma_1+1)x'\right)x'' + 2(\gamma_1-1)x' .$$

In terms of η, (38) becomes

$$(1-\eta)^2 x'' + (1-\eta)x' + x = \left(1-(\gamma_1+1)x'\right)x'' - 2\,\frac{(\gamma_1-1)}{\gamma_1}x' ,$$

or, keeping only the leading terms in η,

$$\left(-2\eta + (\gamma_1+1)x'\right)x'' + \left(\frac{3\gamma_1-2}{\gamma_1}\right)x' = 0 . \tag{41}$$

Let $y = x' - \frac{2}{\gamma_1+1}\eta$. Eq. (41) takes the form

$$y' = -\frac{5\gamma_1-2}{\gamma_1(\gamma_1+1)} - \frac{2(3\gamma_1-2)}{\gamma_1(\gamma_1+1)^2}\,\frac{\eta}{y} .$$

This can be solved by putting $y = \eta Z$, which gives

$$\eta\frac{dZ}{d\eta} = -\frac{(Z+Z_1)(Z+Z_2)}{Z}, \text{ with } Z_1 = \frac{3\gamma_1-2}{\gamma_1(\gamma_1+1)},\ Z_2 = \frac{2}{\gamma_1+1} .$$

Hence

$$\ln\eta = -\int \frac{Z\,dZ}{(Z+Z_1)(Z+Z_2)},$$

$$C\eta = \frac{(Z+Z_1)^{\frac{Z_1}{Z_2-Z_1}}}{(Z+Z_2)^{\frac{Z_2}{Z_2-Z_1}}} , \tag{42}$$

[redacted]

 [redacted]

and

$$\varphi' = 1 + \eta\,(Z_2 + Z). \qquad (43)$$

For small values of $c\eta$

$$Z = -Z_1 + (Z_2 - Z_1)\left[(c\eta)^{\frac{Z_2-Z_1}{Z_1}} - Z_2\,(c\eta)^{2\frac{Z_2-Z_1}{Z_1}} + \ldots\ldots\right].^{*}$$

For $\eta = 0$, $Z = -Z_1$. If c is positive Z increases with increasing η. If $c = (-1)^{\frac{Z_1}{Z_2-Z_1}} a$, $a > 0$, Z varies from $-Z_1$ to $-Z_2$ as η goes from 0 to ∞. This latter case is interesting in explaining a non-uniformity which occurs for very dense walls. A wall of infinite density of course will not move; the appropriate solution is $f = 1, \varphi = \xi, \varphi' = 1$. However it follows from (42), (43) that, independent of the wall density, near $\eta = 0, \varphi' = 1 + \eta\ (Z_2 - Z_1)$, $\frac{\rho}{\rho_1} = 1 - \eta(Z_2 - Z_1)$. Thus the density at the back of the wave always begins to drop linearly, with a slope determined only by γ_1. The explanation is that for large values of a, Z very rapidly shifts from its initial value $-Z_1$ to its final value $-Z_2$ for which the coefficient of the η-dependent term in (43) vanishes. For very dense walls the rapid change in density is thus confined to the very back of the wave; as soon as $a\eta >> 1$, the rate of change in density becomes of order $\eta/(a\eta)^{\frac{Z_2-Z_1}{Z_1}}$.

BEHAVIOR NEAR THE INTERFACE

The numerical integration of (31) and (35) can now be carried out by starting near $\xi = -1$, using the solutions (42), (43) with a trial value of c, and carrying the solution to $\xi = 0$. The solution obtained will be that appropriate to a certain value of $\frac{\rho_1}{\rho_{-1}} \frac{\gamma_1}{(\gamma+1)}$, determined by (36).

One check on the numerical integration can be obtained from the fact that the expansion of the material near the interface takes place when the pressure is low and little work is done, so the temperature should be the same

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$^{*}Z = -Z_1 + (Z_2 - Z_1)^{Z_2/Z_1}(c\eta)^{\frac{Z_2-Z_1}{Z_1}}\left\{1 + \frac{Z_2}{Z_1}\left[(Z_2-Z_1)(c\eta)\right]^{\frac{Z_2-Z_1}{Z_1}} + \cdots\right\}?$

 [redacted]

as in the material not yet reached by the wave. This can be verified analytically from (39), which gives as the leading term near $\xi = 0$, $f(0)\,\varphi'(0) = 1$.

We also have the relations analogous to (17) and (18), from the momentum law

$$f(0) = 1 - 2\gamma_1 \left[2\int \varphi \, d\xi + 1 \right] .$$

from the energy law

$$f(0)\varphi(0) = \frac{3}{\gamma_1 - 1} - 3\int_0^1 \left[\frac{1}{\gamma_1 - 1} f \varphi' + \frac{6_1}{2}\left(\varphi - \varphi'\xi\right)^2 \right] d\xi .$$

[redacted]